Pontryagin Maximum Principle and Applications in Escaping Problem

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Introduction

In this research project, we studied the **Pontryagin Maximum Principle**, which is commonly used in control theory, to find the best possible control and trajectory for a dynamic system. The principle states that it is necessary for any optimal control along with the optimal state trajectory to solve the so-called **Hamiltonian system**, which is a two-point boundary value problem, plus a maximum condition of the control Hamiltonian.

Basic Optimal Control Problem

Considering an ordinary differential equation (ODE)

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \quad (t > 0), \\ \mathbf{x}(0) = x_0, \end{cases} \tag{1}$$

given the initial point $x_0 \in \mathbb{R}^n$ and the function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$. Next, we generalize (1) into system (2) with some function $\alpha : [0, \infty) \to A$ where $A \subset \mathbb{R}^m$, such function α is called a **control**.

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \alpha(t)) \quad (t > 0), \\ \mathbf{x}(0) = x_0. \end{cases} \tag{2}$$

Now, we have $\mathbf{f} : \mathbb{R}^n \times A \to \mathbb{R}^n$ and the control will act on the solution $\mathbf{x}(\cdot)$. We then define the **cost functional**

$$J[\alpha(\cdot)] := \int_0^T r(\mathbf{x}(t), \alpha(t))\, dt + g(\mathbf{x}(T)), \tag{3}$$

where $r : \mathbb{R}^n \times A \to \mathbb{R}$ is called the **running cost**, $g : \mathbb{R}^n \to \mathbb{R}$ is called the **terminal cost**, $T > 0$ is called the **terminal time**, which are all given. The basic optimal control problem is to find the optimal control for the system to *minimize* the cost.

Pontryagin Maximum Principle

To find the optimal control for the problem, we can apply **Pontryagin Maximum Principle** (PMP). Here gives the definition of Hamiltonian in control theory.

Definition 1 *The Hamiltonian is the function*

$$H(x, \lambda, a) := \mathbf{f}(x, a) \cdot \lambda + r(x, a) \quad (x, \lambda \in \mathbb{R}^n, a \in A).$$

Then we can proceed to the main theorem of PMP.

Theorem 1 *For a basic optimal control problem, assume $\alpha^*(\cdot)$ is the optimal control, and $\mathbf{x}^*(\cdot)$ is the corresponding trajectory. Then there exists a function $\lambda^* : [0, T] \to \mathbb{R}^n$, called the* ***costate****, such that*

$$\dot{\mathbf{x}}^*(t) = \nabla_\lambda H(\mathbf{x}^*(t), \lambda^*(t), \alpha^*(t)), \tag{4}$$

$$\dot{\lambda}^*(t) = -\nabla_x H(\mathbf{x}^*(t), \lambda^*(t), \alpha^*(t)), \tag{5}$$

and

$$H(\mathbf{x}^*(t), \lambda^*(t), \alpha^*(t)) = \max_{a \in A} H(\mathbf{x}^*(t), \lambda^*(t), a),$$

where $0 \le t \le T$. In addition, the mapping

$$t \mapsto H(\mathbf{x}^*(t), \lambda^*(t), \alpha^*(t))$$

is constant. Finally, we have the terminal condition

$$\lambda^*(T) = \nabla g(\mathbf{x}^*(T)). \tag{6}$$

Linear Quadratic Form

Linear quadratic form is a special form of control problem, with linear ODE system and quadratic cost functional. Here is an example in linear quadratic form: consider a simple linear dynamics

$$\begin{cases} \dot{x}(t) = x(t) + \alpha(t) \\ x(0) = x_0, \end{cases} \tag{7}$$

with the quadratic cost functional

$$J[\alpha(\cdot)] = \int_0^T x(t)^2 + \alpha(t)^2\, dt, \tag{8}$$

which we want to minimize. In this example, the ODE system (7) is linear and the cost is a quadratic function. Linear quadratic form provides standard derivatives, which is convenient for further derivations and substitutions.

Escaping Problem

Now we apply PMP to a realistic problem. Suppose audience in a cinema need to escape from the cinema theater through exit(s) when an emergency happens. We aim to *minimize* the population cost during the escaping. To simplify the mathematical model, we assume there is no seats so agents can move freely. Initially, agents will choose the nearest exit as their destinations. The control of this problem is the **velocity** of agents. Considering the advantage of **linear quadratic form** mentioned in previous section, we formulate the problem as followings:

The ODE system is

$$\begin{cases} \dot{\mathbf{x}}(t) = \dfrac{d\mathbf{x}(t)}{dt} = \mathbf{v}(t), \\ \mathbf{x}(0) = \mathbf{x}_0. \end{cases} \tag{9}$$

where

$\mathbf{x}(t) = (x_1(t), y_1(t), x_2(t), y_2(t), \cdots, x_n(t), y_n(t))^T \in \mathbb{R}^{2n}$,
$\mathbf{v}(t) = (u_1(t), v_1(t), u_2(t), v_2(t), \cdots, u_n(t), v_n(t))^T \in \mathbb{R}^{2n}$

are positions and velocities of agents at time t,

$$\mathbf{x}_0 = (x_1(0), y_1(0), x_2(0), y_2(0), \cdots, x_n(0), y_n(0))^T$$

is the initial positions. And the cost is defined as

$$\begin{aligned} J(\mathbf{v}(t)) =& \frac{1}{2}\int_0^T [\mathbf{v}^T(t)A\mathbf{v}(t) - \mathbf{x}^T(t)Q\mathbf{x}(t)]\, dt \\ &+ \frac{1}{2}\mathbf{x}^T(T)\mathbf{x}(T) + \mathbf{b}^T\mathbf{x}(T) + K, \end{aligned} \tag{10}$$

where the running cost consists of the energy expense ($\mathbf{v}^T A\mathbf{v}$) and disturbance from other agents, measured by relative distances ($\mathbf{x}^T Q\mathbf{x}$), terminal cost is determined by relative distances between terminal positions and positions of chosen exits for those who did not escape on or before the terminal time. Here, $A, Q \in \mathbb{R}^{2n \times 2n}$ are positive semi-definite matrices, $\mathbf{b} = -(a_1, b_1, \cdots, a_n, b_n)^T \in \mathbb{R}^n$ and $K = \frac{1}{2}\sum_{i=1}^n (a_i^2 + b_i^2)$ are fixed by positions of exits and the choices of agents, where (a_i, b_i) is the position of i-th exit. By PMP, we define the Hamiltonian as

$$H = \mathbf{v}^T A\mathbf{v} - \mathbf{x}^T Q\mathbf{x} + \lambda^T\mathbf{v},$$

where λ is the costate. Then we have

$$\dot{\mathbf{x}}(t) = (\frac{\partial H}{\partial \lambda})^T = \mathbf{v}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{11}$$

$$-\dot{\lambda}(t) = (\frac{\partial H}{\partial \mathbf{x}})^T = -Q\mathbf{x}(t), \quad \lambda(T) = \mathbf{x}(T) + \mathbf{b}, \tag{12}$$

$$\mathbf{0} = \frac{\partial H}{\partial \mathbf{v}} = A\mathbf{v} + \lambda^T. \tag{13}$$

By (13), we know that $\mathbf{v} = -A^{-1}\lambda$.

Now we arrive at a system of ODEs with boundary conditions

$$\begin{cases} \dot{\mathbf{x}} = -A^{-1}\lambda, & \mathbf{x}(0) = \mathbf{x}_0, \\ \dot{\lambda} = Q\mathbf{x}, & \lambda(T) = \mathbf{x}(T) + \mathbf{b}. \end{cases} \tag{14}$$

By using **forward and backward Euler's method** and a substitution $\tau := T - t$ for the terminal condition, we can solve system (14) for approximated solutions.

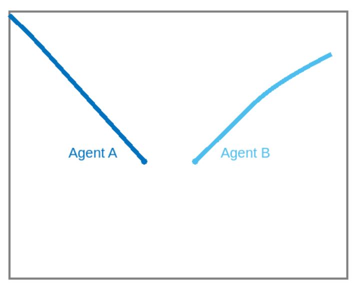
Simulation and Results

We simulate a two-agent case in a cinema theater with size 20×16. After placing the theater in the first quadrant, the locations of two exits of the theater are $(0, 16)$ and $(20, 16)$ respectively. Define $Q = k_3 Q^0$ and

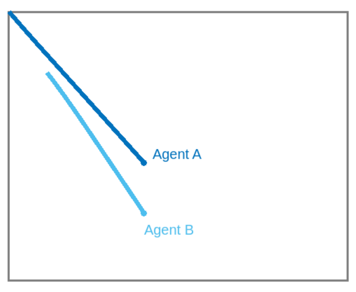
$$Q^0 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_2 \end{pmatrix}.$$

We run the simulations using different $k_1, k_2, k_3, \mathbf{x}_0, T$. Note that $\mathbf{b}$ depends on $\mathbf{x}_0$ and remains the same during the escaping.

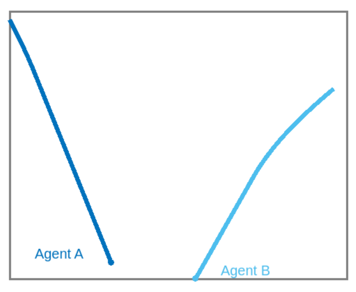
- $k_1 = 1, k_2 = 2, k_3 = 1.$



$\mathbf{x}_0 = (8, 7, 11, 7)^T, T = 1.5,$ $\mathrm{b} = (0,-16,-20,-16)^T$

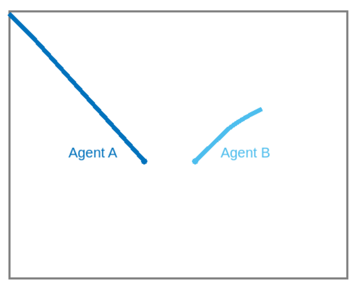


$\mathbf{x}_0 = (8, 7, 8, 4)^T, T = 1.5,$ $\mathrm{b} = (0,-16,0,-16)^T$

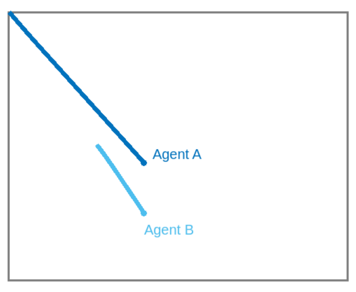


$\mathbf{x}_0 = (6, 1, 11, 0)^T, T = 1.5,$ $\mathrm{b} = (0,-16,-20,-16)^T$

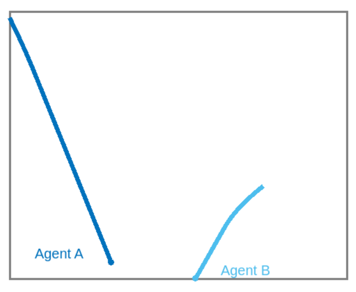
- $k_1 = 1, k_2 = 4, k_3 = 1.$



$\mathbf{x}_0 = (8, 7, 11, 7)^T, T = 1.5,$ $\mathrm{b} = (0,-16,-20,-16)^T$

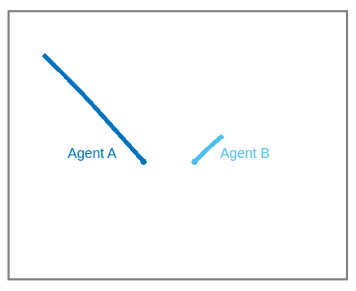


$\mathbf{x}_0 = (8, 7, 8, 4)^T, T = 1.5,$ $\mathrm{b} = (0,-16,0,-16)^T$

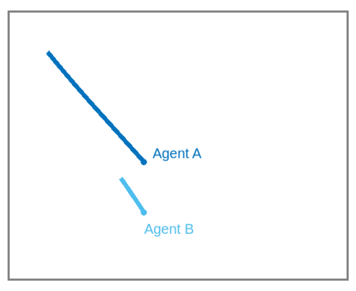


$\mathbf{x}_0 = (6, 1, 11, 0)^T, T = 1.5,$ $\mathrm{b} = (0,-16,-20,-16)^T$

- $k_1 = 1, k_2 = 4, k_3 = 1.$



$\mathbf{x}_0 = (8, 7, 11, 7)^T, T = 0.7,$ $\mathrm{b} = (0,-16,-20,-16)^T$

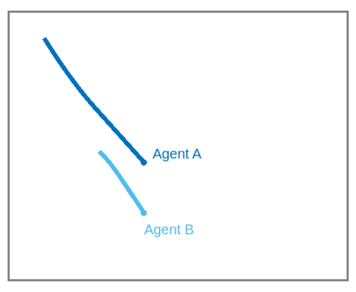


$\mathbf{x}_0 = (8, 7, 8, 4)^T, T = 0.7,$ $\mathrm{b} = (0,-16,0,-16)^T$

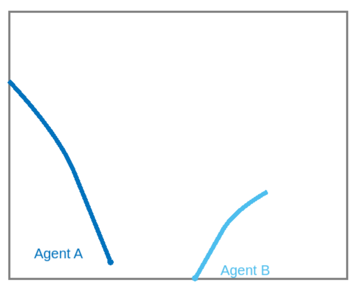
$\mathbf{x}_0 = (6, 1, 11, 0)^T, T = 0.7,$ $\mathrm{b} = (0,-16,-20,-16)^T$

- $k_1 = 1, k_2 = 2, k_3 = 5.$

$\mathbf{x}_0 = (8, 7, 11, 7)^T, T = 0.7,$ $\mathrm{b} = (0,-16,-20,-16)^T$



$\mathbf{x}_0 = (8, 7, 8, 4)^T, T = 0.7,$ $\mathrm{b} = (0,-16,0,-16)^T$



$\mathbf{x}_0 = (6, 1, 11, 0)^T, T = 0.7,$ $\mathrm{b} = (0,-16,-20,-16)^T$

Conclusion and Discussion

In this research, we show the feasibility of PMP in solving the escaping problem in linear quadratic form. Specifically, we simulated a two-agent case in a rectangle theater and visualized the optimal trajectories of agents. In future research work, we can consider the following extensions:

- We may increase the number of agents.
- We may change the optimization objectives.
- We may consider different geometries of the space, so that the escaping problem can be discussed in other public spaces, such as auditoriums and stadiums.
- Instead of assuming the escaping problem as a cooperative game and *minimizing* the population cost, we may solve a **Nash Equilibrium** problem to *minimize* the individual costs. Agents amend their individual strategies given others' choices until an equilibrium is reached.

References

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